View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Topology Vol. 3, pp. 123–135. Pergamon Press, 1965. Printed in Great Britain

HIGHER ORDER WHITEHEAD PRODUCTS

GERALD J. PORTER†

(*Received* 22 *July* 1963)

THE BRACKET product of J. H. C. Whitehead [11] assigns an element $[f, g] \in \pi_{r+t-1}(X)$ to each pair of elements $f \in \pi_r(X)$ and $g \in \pi_t(X)$. Equivalently an element, $W(\varphi) \in \pi_{r+t-1}(X)$, is assigned to each element $\varphi \in \pi(S^r \vee S^t, X)$. In this paper we generalize this product by replacing the wedge of two spheres with a "wedge" of n suspensions. In particular let $T_1(X_1, \ldots, X_n)$ be the subset of the cartesian product $X_1 \times \ldots \times X_n$ consisting of those n-tuples with at least one co-ordinate at a base point. Let

$$\wedge(X_1, \ldots, X_n) = X_1 \times \ldots \times X_n / T_1(X_1, \ldots, X_n).$$

Then to each map $\varphi : T_1(\Sigma A_1, \ldots, \Sigma A_n) \to X$ an element $W(\varphi) \in \pi(\Sigma^{n-1} \wedge (A_1, \ldots, A_n), X)$ is assigned. $W(\varphi)$ is the generalized n^{th} order Whitehead product of φ. The original Whitehead product product is, in this terminology, a 2^{nd} order product.

The contents of the paper are, briefly, as follows. Section 1 contains our notation and the definition of the generalized higher order Whitehead product (GHOWP). The properties of the GHOWP are studied in §2. Section 3 is devoted to the study of the connection between higher order Whitehead products and the cohomology of the range space. As an application of the results of this section it is shown that there exist non-zero Whitehead products of arbitrarily high order in $BSU(n)$ for each n. The appendix contains some auxiliary homotopy theory needed for the definition of the GHOWP.

The material in this paper was contained in the author's doctorial dissertation written at Cornell University under the direction of Professor William Browder. The author also wishes to acknowledge the helpful suggestions of Professor Peter Hilton.

§1. PRELIMINARIES

We shall assume that all spaces are countable connected CW-complexes with base point and all maps are continuous and base point preserving. We denote the category of these spaces and maps by $\mathscr{C}$. $\mathscr{C}^n$ is the category of n-tuples of objects and maps of $\mathscr{C}$. We use several functors which we now describe.

The functors $T_i : \mathscr{C}^n \to \mathscr{C}$, $i = 0, 1, \ldots, n$, are defined by setting $T_i(X_1, \ldots, X_n)$ equal to the subset of $X_1 \times \ldots \times X_n$ consisting of those points with at least i co-ordinates at base points.

† The author was supported by NSF Contract G15984 during part of the period in which this paper was prepared for publication.

Since all maps are base point preserving $f_1 \times \ldots \times f_n$ maps the subset $T_i(X_1, \ldots, X_n)$ to $T_i(Y_1, \ldots, Y_n)$. We denote the restriction of $f_1 \times \ldots \times f_n$ to this subset by $T_i(f_1, \ldots, f_n)$. T_0 is the usual cartesian product functor; T_1 is the so called "fat wedge" and T_{n-1} is the one point union functor. There is a natural transformation of functors $T_i \to T_{i-1}$ for $i = 1, \ldots, n$, given by inclusion. The transformation $T_1 \to T_0$ is denoted by j. The smash product functor, $\wedge : \mathscr{C}^n \to \mathscr{C}$, is the quotient functor T_0/T_1. Let $I = [0, 1]$ with basepoint $* = 1$. The reduced cone functor $C: \mathscr{C} \to \mathscr{C}$ is defined to be $\wedge(I, \ )$. Similarly $\Sigma: \mathscr{C} \to \mathscr{C}$; the reduced suspension functor, is defined to be $\wedge(S^1, \ )$. $\Sigma^n: \mathscr{C} \to \mathscr{C}$ is defined inductively by $\Sigma^n X = \Sigma(\Sigma^{n-1} X)$. Σ and C induce functors $\mathscr{C}^n \to \mathscr{C}^n$ which we shall also denote by Σ and C. In particular $\Sigma(X_1, \ldots, X_n) = (\Sigma X_1, \ldots, \Sigma X_n)$ and $C(X_1, \ldots, X_n) = (CX_1, \ldots, CX_n)$.

Let $p: I \to S^1$ be the quotient map. The natural transformation $P: C \to \Sigma$ is defined by $P(X_1, \ldots, X_n) = (\wedge(p, 1_{X_1}), \ldots, \wedge(p, 1_{X_n}))$. $T_0(P)$ is then a natural transformation $T_0 C \to T_0 \Sigma$. We denote $T_0(P)$ by ρ.

We shall not distinguish notationally between a transformation of functors and the transformation's values on the objects of the category.

Let $Id: \mathscr{C} \to \mathscr{C}$ be the identity functor. There are natural transformations $i_C: Id \to C$ and $i_\Sigma: Id \to \Sigma$ given by $i_C(x) = (0, x)$ and $i_\Sigma(x) = (\frac{1}{2}, x)$.

The functor $Q: \mathscr{C}^n \to \mathscr{C}$ is defined by setting $Q(X_1, \ldots, X_n) = \rho^{-1}(T_1 \Sigma(X_1, \ldots, X_n))$ and $Q(f_1, \ldots, f_n) = T_0 C(f_1, \ldots, f_n) | Q(X_1, \ldots, X_n)$. Let $\bar{\rho} = \rho | Q$. $\bar{\rho}$ is then a natural transformation $Q \to T_1 \Sigma$. Moreover $(\rho, \bar{\rho}): (T_0 C, Q) \to (T_0 \Sigma, T_1 \Sigma)$ is easily seen to be a relative homeomorphism. (A geometric definition of Q is given in the appendix.)

DEFINITION (1.1). *Let F and G be functors $\mathscr{A} \to \mathscr{C}$ for some category $\mathscr{A}$. We say that T is a homotopy equivalent transformation $F \to G$ if for each $A \in \mathscr{A}$, $T(A): F(A) \to G(A)$ is a homotopy equivalence and for each map $h \in \mathscr{A}$, $h: A \to B$, $T(B)F(h) \sim G(h)T(A)$. We say that F is homotopy equivalent to G if there exists a homotopy equivalent transformation $F \to G$.*

The following theorem is proven in the appendix.

THEOREM (1.2). *There exists a homotopy equivalent transformation $\bar{h}: \Sigma^{n-1} \wedge \to Q$ (where the two functors have domain category $\mathscr{C}^n$).*

As usual we let $\pi(X, Y)$ stand for the set of homotopy classes of maps X to Y and $\{f\}$ stand for the homotopy class of the map f.

DEFINITION (1.3). *Given $\varphi: T_1 \Sigma(A_1, \ldots, A_n) \to X$, $n \geqslant 2$, we define*

$$W(\varphi) \in \pi(\Sigma^{n-1} \wedge (A_1, \ldots, A_n), X)$$

the n^{th} order generalized Whitehead product, to be $\varphi_* \{\bar{\rho} \bar{h}\}$. φ_* is the induced map in homotopy and hence it is immediate that $W(\varphi)$ depends only upon the homotopy class of φ.

(It will be understood throughout the paper that "n^{th} order generalized Whitehead product" means $n \geqslant 2$.)

We say that $\varphi: T_i \Sigma(A_1, \ldots, A_n) \to X$, $i < n$, is of type $(f_1, \ldots, f_n)$ or more briefly $\varphi \in (f_1, \ldots, f_n)$ if $\varphi k_j \sim f_j$ for $j = 1, \ldots, n$, where $k_j: \Sigma A_j \to T_i \Sigma(A_1, \ldots, A_n)$ is the canonical injection.

DEFINITION (1.4). *The set of* n^{th} *order Whitehead products of type* $(f_1, \ldots, f_n)$ *is denoted* $[f_1, \ldots, f_n]$ *and defined by* $[f_1, \ldots, f_n] = \{W(\varphi) | \varphi : T_1\Sigma(A_1, \ldots, A_n) \to X,\ \varphi \in (f_1, \ldots, f_n)\}$.

We stress the fact that $W(\varphi)$ is a well defined element while $[f_1, \ldots, f_n]$ is a subset (perhaps empty) of $\pi(\Sigma^{n-1}\wedge(A_1, \ldots, A_n), X)$.

If the A_i's are all spheres then $[f_1, f_2]$ is the original Whitehead product. $[f_1, f_2, f_3]$ is, in this case, the Zeeman product studied by Hardie [5]. Hardie [4] has also given the definition of $[f_1, \ldots, f_n]$ when the A_i's are all spheres. When the A_i's are arbitrary, $[f_1, f_2]$ is the "generalized Whitehead product" studied by Arkowitz [1].

We shall let HOWP stand for the case in which all the A_i's are spheres (as opposed to GHOWP). In §2 $W(\varphi)$ may stand for either a homotopy class or a map in the homotopy class. The meaning will be clear from the context.

§2. PROPERTIES OF THE GHOWP

THEOREM (2.1). (*Naturality*) *Let* $f_i : A_i \to B_i$, $h_i : B_i \to X$, $i = 1, \ldots, n$, $g : X \to Y$ *and* $\varphi : T_1\Sigma(B_1, \ldots, B_n) \to X$, *then*

(a) $(\Sigma^{n-1}\wedge(f_1, \ldots, f_n))^* W(\varphi) = W(\varphi T_1\Sigma(f_1, \ldots, f_n))$

(b) $g_* W(\varphi) = W(g\varphi)$

(c) $(\Sigma^{n-1}\wedge(f_1, \ldots, f_n))^*[h_1, \ldots, h_n] \subset [h_1(\Sigma f_1), \ldots, h_n(\Sigma f_n)]$

(d) $g_*[h_1, \ldots, h_n] \subset [gh_1, \ldots, gh_n]$.

Proof. Parts (a) and (c) follow from the fact that $\bar{\rho}$ is a natural transformation and $\bar{h}$ is a homotopy equivalent transformation. Parts (b) and (d) follow since $(g\varphi)_* = g_*\varphi_*$.

THEOREM (2.2). *Let* $j : T_1\Sigma \to T_0\Sigma$ *be the natural transformation given by inclusion, then* $W(j) = 0$.

Proof. T_0C is a contractible functor, i.e. there exists a natural transformation $H : CT_0C \to T_0C$ such that $Hi_C = 1$. Let $h : C\Sigma^{n-1}\wedge \to T_0C$ be the following composition

$$C\Sigma^{n-1}\wedge \xrightarrow{C\bar{h}} CQ \longrightarrow CT_0C \xrightarrow{H} T_0C$$

where the middle transformation is inclusion.

Clearly $hi_C = \bar{h}$. The pair $(h\rho, \bar{h}\bar{\rho})$ determines an element α of the relative homotopy group $\pi_n(\wedge, j)$ (see [6] for notation) such that $\partial\alpha = \{\bar{h}\bar{\rho}\}$. Thus $W(j) = j_*\partial\alpha = 0$ by exactness.

Remark. α is a generalization of the star product of Blakers and Massey [2]. If $A_i = S^{m_i - 1}$, α is a generator of the relative homotopy group $\pi_N(T_0\Sigma, T_1\Sigma) = Z$ where $N = \sum_{i=1}^{n} m_i$. Starting with α we could have defined $W(\varphi)$ to be $\varphi_*\partial\alpha$.

Let $i : T_1 \to T_1$ be the identity transformation. Consider the map

$$T_1\Sigma \smile_{W(i)} C\Sigma^{n-1}\wedge : \mathscr{C}^n \to \mathscr{C}$$

defined in the obvious way. Theorem (2.1a) implies that this map is a functor.

THEOREM (2.3). *There is a homotopy equivalent transformation*

$$\lambda : T_1\Sigma \smile_{W(i)} C\Sigma^{n-1}\wedge \to T_0\Sigma$$

such that

$$\begin{tikzcd}
T_1\Sigma \smile_{W(i)} C\Sigma^{n-1} \arrow[r, "\lambda"] & T_0\Sigma \\
T_1\Sigma \arrow[u, "k"] \arrow[r, "i"] & T_1\Sigma \arrow[u, "j"']
\end{tikzcd}$$

commutes, where k is inclusion.

Proof. Since $(\rho, \bar{\rho}):(T_0C, Q) \to (T_0\Sigma, T_1\Sigma)$ is a relative homeomorphism for each object in $\mathscr{C}^n$, there is a natural transformation $\eta : T_1\Sigma \smile_{\bar{\rho}} T_0C \to T_0\Sigma$ which is a homeomorphism for each object in $\mathscr{C}^n$ and such that $\eta | T_1\Sigma = j$. Set λ equal to

$$\eta(i \smile h) : T_1\Sigma \smile_{W(i)} C\Sigma^{n-1}\wedge \to T_0\Sigma.$$

It is clear that this is a homotopy equivalent transformation and $\lambda k = \eta(i \smile h)k = j$.

COROLLARY. *There exists $\mu : T_0\Sigma \to T_1\Sigma \smile_{W(i)} C\Sigma^{n-1}\wedge$ such that $\mu j = k$* and *μ is a homotopy equivalent transformation.*

THEOREM (2.4). *$\varphi : T_1\Sigma(A_1, \ldots, A_n) \to X$ can be extended to $\psi : T_0\Sigma(A_1, \ldots, A_n) \to X$ if and only if $W(\varphi) = 0$.*

Proof. Assume that ψ exists, then $\varphi = \psi j$. $W(\varphi) = W(\psi j) = \psi_* W(j) = 0$ by Theorem (2.2). If $W(\varphi) = 0$, $\varphi W(i)$ is null homotopic and φ can be extended to

$$\bar{\varphi} : (T_1\Sigma \smile_{W(i)} C\Sigma^{n-1}\wedge)(A_1, \ldots, A_n) \to X.$$

Let $\psi = \bar{\varphi}\mu : T_0\Sigma(A_1, \ldots, A_n) \to X$. Since $\bar{\varphi}\mu j = \bar{\varphi}k = \varphi$, ψ extends φ.

THEOREM (2.5). *Let $\varphi : T_1\Sigma(A_1, \ldots, A_n) \to X$. If X is an H-space, $W(\varphi) = 0$.*

Proof. By the previous theorem it suffices to show that φ can be extended to $T_0\Sigma$. Hilton has shown that $\Sigma T_1\Sigma$ is a retract of $\Sigma T_0\Sigma$ for each object in $\mathscr{C}^n$. Hence $\Sigma\varphi$ can be extended to $\psi : \Sigma T_0\Sigma \to \Sigma X$. ψ induces $\psi' : T_0\Sigma \to \Omega\Sigma X$ in the usual way. James has shown that if X is an H-space there exists $r : \Omega\Sigma X \to X$ such that $rk \sim 1$ where $k : X \to \Omega\Sigma X$ is given by $k(x)(t) = (t, x)$. Thus we have $r\psi' : T_0\Sigma \to X$. $r\psi' j = rk\varphi \sim \varphi$. Hence φ has an extension to $T_0\Sigma$.

COROLLARY (1). *Let $f_i : \Sigma A_i \to X$ for $i = 1, \ldots, n$. If X is an H-space $[f_1, \ldots, f_n] = 0$.*

COROLLARY (2). *The induced map in homotopy*

$$(W(\varphi))^* : \pi(X, H) \to \pi(\Sigma^{n-1}\wedge(A_1, \ldots, A_n), H)$$

is zero if H is an H-space.

Proof. Let $f \in \pi(X, H)$. Then $(W(\varphi))^*(f) = W(f\varphi) = 0$ by Theorem (2.5), since H is an H-space.

In particular if $H = K(G, n)$, $n > 0$, we have

COROLLARY (3). *$(W(\varphi))^* : H^n(X; G) \to H^n(\Sigma^{n-1}\wedge(A_1, \ldots, A_n); G)$ is the zero map.*

ROLLARY (4). *If Σ_* is the suspension homomorphism, then $\Sigma_* W(\varphi) = 0$.*

Proof. $\Sigma_* W(\varphi) = 0$ if and only if $k_* W(\varphi) = 0$, where k is as in the proof of Theorem (2.5). $k_* W(\varphi) = W(k\varphi) = 0$ since the range of $k\varphi$ is $\Omega\Sigma X$, an H-space.

There are theorems analogous to Theorems (2.3) and (2.4) for the functor $T_i\Sigma$ and maps $\varphi: T_i\Sigma(A_1, \ldots, A_n) \to X$. We first define several functors and natural transformations. We shall assume that n is a fixed integer for the next paragraphs.

Let S_i be the set of order preserving functions $[1, 2, \ldots, n-i+1] \to [1, 2, \ldots, n]$. For each $\sigma \in S_i$ we define the functor $\sigma: \mathscr{C}^n \to \mathscr{C}^{n-i+1}$ by $\sigma(A_1, \ldots, A_n) = (A_{\sigma(1)}, \ldots, A_{\sigma(n-i+1)})$. $j_\sigma: T_1\Sigma\sigma \to T_i\Sigma$ and $k_\sigma: T_0\Sigma\sigma \to T_{i-1}\Sigma$ are the natural transformations given by injection. Let $\theta: \bigvee_{\sigma \in S_i} \Sigma^{n-i}\wedge\sigma \to T_i\Sigma$ be given by $\theta|\Sigma^{n-i}\wedge\sigma = W(j_\sigma)$.

THEOREM (2.6). *The following functors $\mathscr{C}^n \to \mathscr{C}$ are homotopy equivalent*

(i) $T_{i-1}\Sigma$

(ii) $T_i \smile_\theta C \bigvee_{\sigma \in S_i} \Sigma^{n-i}\wedge\sigma$

(iii) $T_i \smile_\theta \bigvee_{\sigma \in S_i} C\Sigma^{n-i}\wedge\sigma$

Proof. Since $C\vee$ and $\vee C$ are homotopy equivalent (ii) and (iii) are homotopy equivalent. We observe that $\bigcup_{\sigma \in S_i} k_\sigma(T_0\Sigma\sigma) = T_{i-1}\Sigma$ and $\bigcup_{\sigma \in S_i} j_\sigma(T_1\Sigma\sigma) = T_i\Sigma$. It follows that (iii) and (i) are homotopy equivalent since $T_0\Sigma\sigma$ is homotopy equivalent to $T_1\Sigma\sigma \smile_{W(1)} C\Sigma^{n-i}\wedge\sigma$ by Theorem (2.3).

COROLLARY. *cat* $T_i\Sigma(A_1, \ldots, A_n) \le n - i + 1$, *where cat X stands for the Lusternik–Schnirelmann category of X.*

Proof. The proof is by induction using the fact that attaching a cone to a space increases cat by at most one.

It is easily seen that the method used in the proof of Theorem (2.4) together with Theorem (2.6) gives:

THEOREM (2.7). *$\varphi: T_i\Sigma(A_1, \ldots, A_n) \to X$ can be extended to $T_{i-1}\Sigma(A_1, \ldots, A_n)$ if and only if $W(\varphi j_\sigma) = 0$ for each $\sigma \in S_i$.*

Using Theorem (2.7) we see that for $n > 2$, $[f_1, \ldots, f_n]$ is non-empty if and only if certain i^{th} order products are zero for $2 \le i < n$. In this sense the n^{th} order product is an $(n-1)$-ary operation.

Let $(\Sigma A)_n$ be the n^{th} reduced product space of ΣA [8]. Let A^n stand for the n-tuple $(A, \ldots, A)$. Then there is a natural transformation induced by projection $p_i: T_i\Sigma(A^n) \to (\Sigma A)_n$ such that the following diagram commutes

$$\begin{array}{ccc} T_i\Sigma(A^n) & \longrightarrow & T_{i-1}\Sigma(A^n) \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_{i-1}} \\ (\Sigma A)_{n-i} & \longrightarrow & (\Sigma A)_{n-i+1} \end{array}$$

when the horizontal lines are inclusion transformations.

THEOREM (2.8). *The following are equivalent*

(i) $\varphi:(\Sigma A)_{n-1} \to X$ *can be extended to* $(\Sigma A)_n$

(ii) $\varphi p_i : T_i\Sigma(A^{n+i-1}) \to X$ *can be extended to* $T_{i-1}\Sigma(A^{n+i-1})$

(iii) $W(\varphi p_1) = 0$

Proof. By Theorem (2.7), φp_i can be extended to $T_{i-1}\Sigma(A^{n+i-1})$ if and only if $W(\varphi p_i j_\sigma) = 0$ for each $\sigma \in S_i$. Since $p_i j_\sigma = p_1$ for each $\sigma \in S_i$ (ii) is equivalent to (iii). We now show (i) equivalent to (iii). If $\psi:(\Sigma A)_n \to X$ extends φ then ψp_{i-1} extends φp_i, and hence, by the above, $W(\varphi p_1) = 0$. Now assume $W(\varphi p_1) = 0$. We recall that $(\Sigma A)_n = (\Sigma A)_{n-1} \smile_{p_1} T_0\Sigma(A^n)$. Since $W(\varphi p_1) = 0$, φp_1 has an extension $\theta: T_0\Sigma(A^n) \to X$. The induced map $\varphi \smile \theta: (\Sigma A)_{n-1} \smile_{p_1} T_0\Sigma(A^n) \to X$ is then well defined and extends φ.

DEFINITION (2.9). *A map* $\varphi: T_i\Sigma(A^n) \to X$ *is said to be reducible if there exists a map* $\psi:(\Sigma A)_{n-i} \to X$ *such that* $\psi p_i \sim \varphi$.

Remark. Each map $\psi:(\Sigma A)_n \to X$ determines reducible maps $\varphi_i: T_i\Sigma(A^{n+i}) \to X$ for $i \geq 0$ defined by $\varphi_i = \psi p_i$.

COROLLARY (TO THEOREM (2.8)). *If a reducible map* $\varphi: T_i\Sigma(A^n) \to X$ *may be extended to* $T_{i-1}\Sigma(A^n)$ *then the extension may be chosen to be reducible.*

Proof. Suppose $\varphi \sim \psi p_i$. Since φ is extendable there is an extension θ of ψp_i. Then by Theorem (2.8), there is an extension, $\bar{\psi}$, of ψ. Hence there is an extension of φ which is homotopic to $\bar{\psi} p_{i-1}$ and hence reducible.

In general the GHOWP is not additive. A special addition is defined on certain compatible elements. This addition and the previous results on reducible maps are then used to show that certain sets of n^{th} order Whitehead products are non-empty.

DEFINITION (2.10). $\psi, \varphi: T_1\Sigma(A_1, \ldots, A_n) \to X$ *are said to be compatible off the* i^{th} *co-ordinate if*

$$\psi | T_0\Sigma(A_1, \ldots, A_{i-1}, *, A_{i+1}, \ldots, A_n) = \varphi | T_0\Sigma(A_1, \ldots, A_{i-1}, *, A_{i+1}, \ldots, A_n).$$

Two homotopy classes are said to be compatible if they have compatible representatives.

DEFINITION (2.11). *If* ψ *and* $\varphi: T_1\Sigma(A_1, \ldots, A_n) \to X$ *are compatible off the* i^{th} *co-ordinate and* A_i *is a suspension,* $(A_i = \Sigma A_i')$, *we define* $\psi *_i \varphi$ *by:*

$$\psi *_i \varphi((t_1, a_1), \ldots, (t_i, (u, a_i')), \ldots, (t_n, a_n)) = \begin{cases} \psi((t_1, a_1), \ldots, (t_i, (2u, a_i')), \ldots, (t_n, a_n)) & 0 \leq u \leq \frac{1}{2} \\ \varphi((t_i, a_1), \ldots, (t_i, (2u-1, a_i')), \ldots, (t_n, a_n)) & \frac{1}{2} \leq u \leq 1 \end{cases}$$

DEFINITION (2.12). *If* φ *and* A_i *are as in Definition* (2.11) *we define*

$$-_i\varphi((t_1, a_1), \ldots, (t_i, (u, a_i')), \ldots, (t_n, a_n)) = \varphi((t_1, a_1), \ldots, (t_i, (1-u, a_i')), \ldots, (t_n, a_n))$$

THEOREM (2.13). (a) $W(\psi *_i \varphi) = W(\psi) + W(\varphi)$

(b) $W(-_i\varphi) = -W(\varphi)$.

We shall only give a sketch of the proof here. The details are contained in the author's thesis.

Proof. Since A_i is a suspension there is a suspension structure $\gamma: A_i \to A_i \vee A_i$. The map $\Sigma^{n-1}\wedge(1, \ldots, \gamma, \ldots, 1)$ then gives an H'-structure on $\Sigma^{n-1}\wedge(A_1, \ldots, A_n)$ which induces the group addition in $\pi(\Sigma^{n-1}\wedge(A_1, \ldots\ A_n), X)$. An examination of the map $\bar{h}$ (see §1 and the Appendix) shows that $(\psi\bar{p}\bar{h} \vee \varphi\bar{p}\bar{h})\Sigma^{n-1}\wedge(1, \ldots, \gamma, \ldots, 1) \sim (\psi *_i \varphi)\bar{p}\bar{h}$. The details of this proof are somewhat complicated in general since $\bar{h}$ is not given explicitly. The interested reader can, however, check the details for the case in which the A_i's are spheres. The above homotopy proves part (a). Part (b) follows similarly.

We shall let (f^k) stand for $(f, \ldots, f)$, k times, and $[f^k]$ stand for $[f, \ldots, f]$, k times.

THEOREM (2.14). *Let $\varphi: T_1\Sigma(A^k) \to X$ be a reducible map of type (f^k). If A is a suspension there exists a reducible map ψ of type $((\alpha f)^k)$ for any integer α such that $W(\psi) = \alpha^k W(\varphi)$.*

Proof. Let

$$\psi_1 = \varphi *_1 \varphi *_1 \ldots *_1 \varphi \qquad (\alpha \text{ times})$$
$$\psi_2 = (\psi_1) *_2 (\psi_1) *_2 \ldots *_2 (\psi_1) \qquad (\alpha \text{ times})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$\psi = (\psi_{k-1}) *_k (\psi_{k-1}) *_k \ldots *_k (\psi_{k-1}) \qquad (\alpha \text{ times})$$

It follows from Theorem (2.13) that $W(\psi) = \alpha^k W(\varphi)$. It is easily seen by writing out ψ explicitly in terms of φ that ψ is a reducible map.

THEOREM (2.15). *Let A be a suspension and let $f: \Sigma A \to X$. If $\pi(\Sigma^{k-1}\wedge(A^k), X)$ is a finite group of order prime to p for each k, $2 \le k < r$, then for each $k \le r$ there is an integer $\alpha_k \not\equiv 0 \pmod p$ such that $[(\alpha_k f)^k]$ is non-empty.*

Proof. The case $k = 2$ is trivial since $[f, f]$ is defined. Assume that a reducible map $\varphi_{k-1}: T_1\Sigma(A^{k-1}) \to X$ of type $((\alpha_{k-1}f)^{k-1})$ has been defined with $\alpha_{k-1} \not\equiv 0 \pmod p$. $W(\varphi_{k-1}) \in \pi(\Sigma^{k-2}\wedge(A^{k-1}), X)$ and hence is of finite order q prime to p. Let β be the smallest integer such that $\beta^{k-1} \equiv 0 \pmod q$. Clearly $\beta \not\equiv 0 \pmod p$. Let $\alpha_k = \alpha_{k-1}\beta$. Then $\alpha_k \not\equiv 0 \pmod p$. By Theorem (2.14) there exists a reducible map $\psi_{k-1}: T_1\Sigma(A^{k-1}) \to X$ of type $((\alpha_k f)^{k-1})$ such that $W(\psi_{k-1}) = \beta^{k-1}W(\varphi_{k-1}) = 0$. By the Corollary to Theorem (2.8) there is a reducible extension of ψ_{k-1} to $T_0\Sigma(A^{k-1})$. This induces a reducible map $\varphi_k: T_1\Sigma(A^k) \to X$ of type $((\alpha_k f)^k)$. Hence $[(\alpha_k f)^k]$ is non-empty.

Application. Let $BSU(n)$ be the classifying space of $SU(n)$. Let $f: S^{2n} \to BSU(n)$ be given. Let p by any prime greater than or equal to n. Let $r = \left[\frac{p}{n}\right] + 1$ where [ ] is the "greatest integer in" function. Then we will show that there exists an integer $\alpha \not\equiv 0 \pmod p$ such that $[(\alpha f)^r]$ is non-empty. By Theorem (2.15) it suffices to show that $\pi_{2ni-1}(BSU(n))$ is of finite order prime to p for $2 \le i < r$. Let $G|_p$ stand for the p component of the finite group G.

James [9] has shown that $\pi_{2j}(SU(n))$ is finite. We now show

LEMMA. $\pi_{2j}(SU(n))|_p = 0$ *for* $j < p$.

Proof. $\pi_{2j}(SU(1)) = 0$ for all j. We proceed by induction on n. Assume true for $n - 1$. The exact homotopy sequence of the fibration $SU(n-1) \to SU(n) \to S^{2n-1}$ gives

$$\to \pi_{2j}(SU(n-1))|_p \to \pi_{2j}(SU(n))|_p \to \pi_{2j}(S^{2n-1})|_p \to$$

since $\pi_{2j}(S^{2n-1})$ is finite. $\pi_{2j}(SU(n-1))|_p = 0$ by induction. Corollary (9.3) [7, p. 320] states that $\pi_{2j}(S^{2n-1})|_p = 0$ if $2j < 2n - 1 + 2p - 3$. Since $p \geq j + 1$, this is the case. Hence the above sequence reduces to

$$0 \to \pi_{2j}(SU(n))|_p \to 0.$$

By exactness $\pi_{2j}(SU(n))|_p = 0$.

Our application follows by using the isomorphism between $\pi_{2ni-1}(BSU(n))$ and $\pi_{2ni-2}(SU(n))$ and noticing that since $i < r, i \leq r - 1 \leq \frac{p}{n}$. Thus $ni \leq p$ and $ni - 1 < p$.

We shall show in the next section that for proper choice of f, $0 \notin [(\alpha f)^r]$.

We close this section by remarking that Hardie [5] has computed the modulus of the third order Whitehead product. By using the $*$ addition of this section we have generalized his result to the generalized third order Whitehead product. The determination of the modulus of the higher order Whitehead products remains an open problem.

§3. HOWP'S DISTINGUISHED BY COHOMOLOGY

In this section we study higher order Whitehead products of maps $\varphi: T_1(S^{n_1}, \ldots, S^{n_k}) \to X$ and give conditions on the Steenrod algebra of X which yield non-zero Whitehead products. Throughout this section cohomology will be taken with Z_p coefficients for some prime p. We shall assume that the ring $\tilde{H}^*(X)$ has a minimum set of generators, x_i of dimension m_i, $i = 1, \ldots, M$. Furthermore, to simplify a calculation, we require that $m_i \neq m_j$ if $i \neq j$. In general this restriction may be replaced by weaker restrictions.† We shall write xy to mean $x \smile y$ and shall assume that all products $x_{i_1} \ldots x_{i_n}$ are written canonically with $i_1 \leq i_2 \leq \ldots \leq i_n$. $(H^*(X))^k$ is the ideal of $H^*(X)$ generated by k fold products of positive dimensional elements of $H^*(X)$.

DEFINITION (3.1). *Let Φ be a natural cohomology operation which vanishes in the cohomology of the cartesian product of spheres and let $y \in H^q(X)$. We say the pair (Φ, y) distinguishes the product $x_{k_1} \ldots x_{k_n}$ if*

$$\Phi(y) = \sum \alpha(j_1, \ldots, j_n) x_{j_1} \ldots x_{j_n} \qquad \mathrm{mod}(H^*(X))^{n+1}$$

with $\alpha(k_1, \ldots, k_n) \neq 0$.

DEFINITION (3.2). *We say $x \in H^m(X)$ signifies $f: S^m \to X$ if $f^*(x) \neq 0$.*

THEOREM (3.3). *Let x_{k_i} signify $f_i: S^{m_{k_i}} \to X$ for $i = 1, \ldots, n$. If (Φ, y) distinguishes $x_{k_1} \ldots x_{k_n}$ and no p of the m_{k_i}, $i = 1, \ldots, n$, are equal then $0 \notin [f_1, \ldots, f_n]$.*

Proof. If $[f_1, \ldots, f_n]$ is empty we are done. Hence we may assume there is $W(\varphi) \in [f_1, \ldots, f_n]$. Let $T_0 = T_0(S^{m_{k_1}}, \ldots, S^{m_{k_n}})$. If $W(\varphi) = 0$, by Theorem (2.4), there is a map $\psi: T_0 \to X$ which extends φ. Consider the following diagram.

$$\begin{array}{ccc} H^q(X) & \xrightarrow{\psi^*} & H^q(T_0) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi = 0} \\ H^N(X) & \xrightarrow{\psi^*} & H^N(T_0) \end{array}$$

† In our applications $H^*(X)$ will always be a polynomial ring, $Z_p[x_1, \ldots, x_M]$, dim $x_i \neq$ dim x_j if $i \neq j$.

where $y \in H^q(X)$ and $N = \sum_{i=1}^{n} m_{k_i}$. The diagram commutes by the naturality of Φ. Since $\Phi = 0$ in $H^*(T_0)$, $\Phi\psi^*(y) = 0$. We show below that $\psi^*\Phi(y) \neq 0$. This contradicts our original assumption and hence $W(\varphi) \neq 0$. Since $W(\varphi)$ is an arbitrary element of $[f_1, \ldots, f_n]$ this proves the theorem.

By the naturality of the cup product $\psi^* : (H^*(X))^r \to (H^*(T_0))^r$. Since $(H^*(T_0))^{n+1} = 0$, $\psi^*(H^*(X))^{n+1} = 0$. Thus $\psi^*\Phi(y) = \sum \alpha(j_1, \ldots, j_n)\psi^*(x_{j_1}, \ldots, x_{j_n})$.

To compute $\psi(x_{j_1} \ldots x_{j_n})$ we first notice that

$$\psi^*(x_{j_i}) = \sum 1 \otimes \ldots \otimes f_t^*(x_{j_i}) \otimes \ldots \otimes 1 \qquad \operatorname{mod}(H^*(T_0))^2 \tag{3.4}$$

where the sum is taken over $\{t : k_t = j_i\}$.†

If $(j_1, \ldots, j_n) \neq (k_1, \ldots, k_n)$ there is some i such that $j_i \neq k_t$ for $1 \leq t \leq n$. It follows from (3.4) that $\psi^*(x_{j_i}) \in (H^*(T_0))^2$. Thus $\psi^*(x_{j_1} \ldots x_{j_n}) \in (H^*(T_0))^{n+1} = 0$.

Let K be the set of distinct k_i. For each $k_i \in K$ let v_i be the number of $k_j = k_i$, $1 \leq j \leq n$. M is defined to be $\prod(v_i!)$, where the product is taken over those i such that $k_i \in K$. If no p of the k_i are equal it is clear that $M \neq 0 \pmod p$.

Since $(H^*(T_0))^{n+1} = 0$ it follows from (3.4) that

$$\psi^*(x_{k_1} \ldots x_{k_n}) = \prod_{i=1}^{n} (\sum 1 \otimes \ldots \otimes f_t^*(x_{k_i}) \otimes \ldots \otimes 1).$$

Using the fact that if $i \neq j$, $m_i \neq m_j$, an elementary computation shows

$$\psi^*(x_{k_1} \ldots x_{k_n}) = M f_1^*(x_{k_1}) \otimes \ldots \otimes f_n^*(x_{k_n}).$$

Since x_{k_i} signifies f_i, $f_i^*(x_{k_i}) = \gamma_i s_i$, where s_i is a generator of $H^{m_{k_i}}(S^{m_{k_i}})$ and $\gamma_i \neq 0$. Combining the above results we have

$$\begin{aligned}\psi^*\Phi(y) &= \alpha(k_1, \ldots, k_n)\psi^*(x_{k_1} \ldots x_{k_n}) \\ &= \alpha(k_1, \ldots, k_n)M\gamma_1 \ldots \gamma_n s_1 \otimes \ldots \otimes s_n\end{aligned}$$

which is non-zero since each of the coefficients is non zero. This completes the proof of the theorem.

Remark. We notice that Steenrod operations are natural and vanish in the cohomology of the cartesian product of spheres.

Application. Borel and Serre [3] have shown that $H^*(BSU(k)) = Z_p[x_2, \ldots, x_k]$ where $\dim x_i = 2i$. For $k \geq n$ and $p \geq n$, Serre [10] has shown that there exists a map $g : S^{2n} \to BSU(n)$ such that x_n signifies g. It was shown in §2 that there is an integer $\alpha \neq 0 \pmod p$ such that $[(\alpha g)^r]$ is non-empty, where $r = \left[\dfrac{p}{n}\right] + 1$. Let $i : BSU(n) \to BSU(k)$, for $k \geq n$, be inclusion and let $f = \alpha i g$. x_n signifies f since $f^*(x_n) = (\alpha i g)^* x_n = \alpha g^* x_n \neq 0$. $[f^r]$ is non-empty since $i_*[(\alpha g)^r] \subset [f^r]$.

A direct calculation using the results of Borel and Serre [3] shows that $\mathscr{P}_p^1 x_{rn-p+1}$ distinguishes $x_n^r \pmod p$ in $BSU(k)$ for $n \leq k < n + n/\left[\dfrac{p}{n}\right]$. Thus Theorem (3.3) implies

† We shall not distinguish between $H^*(T_0)$ and $H^*(S^{m_{k_1}}) \otimes \ldots \otimes H^*(S^{m_{k_n}})$.

$0 \notin [f^r]$. Since $[f^r]$ is non-empty, there exists a non-zero r^{th} order Whitehead product in $\pi_{2nr-1}(BSU(k))$ for $n \leq k < n + n/\left[\frac{p}{n}\right]$.

It is clear that p divides the (group) order of the element described above. Using the isomorphism $\Delta : \pi_{2nr-1}(BSU(k)) \to \pi_{2nr-2}(SU(k))$, we see that for some integer β, $\beta \Delta [f, \ldots, f]$ is a non-zero element of (group) order p in $\pi_{2nr-2}(SU(k))$. We extend the usual terminology and refer to $\Delta [f, \ldots, f]$ as a higher order Samelson product of the map Δf.

We notice the following curious fact. If $n = k = 2$, $SU(2) = S^3$ and if p is an odd prime $2nr - 2 = 2p$. Thus $\alpha \Delta [f, \ldots, f]$ is an element of (group) order p in $\pi_{2p}(S^3)$. This is the smallest dimensional homotopy group of S^3 which contains elements of order p. Hence the initial elements of order p in $\pi_*(S^3)$ are generated by a multiple of a higher order Samelson product.

We remark that the existence of non-zero higher order Whitehead products on spheres is still an open problem.

APPENDIX. CONSTRUCTIONS HOMOTOPIC TO THE JOIN

We continue the notation of §1 with the following addition: $K : \mathscr{C} \to \mathscr{C}$ the unreduced cone functor is defined to be the quotient, $T_0(I, \ )/T_0(1, \ )$. There is a natural transformation $\pi : K \to C$ which is induced by the quotient map.

Definition (1). The iterated join functor $J : \mathscr{C}^n \to \mathscr{C}$ is defined inductively as follows. $J(X_1, X_2)$ is the quotient space obtained from $X_1 \times X_2 \times I$ by factoring out the relations: $(x_1, x_2, 0) \sim (x_1, x_2', 0)$ for all $x_2, x_2' \in X_2$ and $(x_1, x_2, 1) \sim (x_1', x_2, 1)$ for all $x_1, x_1' \in X_1$. The base point of $J(X_1, X_2)$ is $(*, *, \frac{1}{2})$. Assume inductively, that the $n - 1$ fold join has been defined. $J(X_1, \ldots, X_n)$ is then defined to be $J(J(X_1, \ldots, X_{n-1}), X_n)$. If $f_i : X_i \to Y_i$ for $i = 1, \ldots, n$, $J(f_1, f_2)(x_1, x_2, t) = (f_1(x_1), f_2(x_2), t)$ by definition and $J(f_1, \ldots, f_n)$ is defined inductively to be $J(J(f_1, \ldots, f_{n-1}), f_n)$.

It is easily checked that J is indeed a functor. The image of J lies in $\mathscr{C}$ since $\mathscr{C}$ is closed under cartesian products.

Arkowitz [1, p. 11] shows that $J(X_1, X_2)$ is homeomorphic to $KX_1 \times X_2 \smile X_1 \times KX_2$. This homeomorphism is given by

$$v(x_1, x_2, t) = \begin{cases} (x_1, (x_2, 1 - 2t)) & 0 \leq t \leq \frac{1}{2} \\ ((x_1, 2t - 1), x_2) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This result may be generalized as follows.

THEOREM (1). *Let C_1 and C_2 be any contractible spaces containing X_1 and X_2 respectively as subcomplexes. Then $J(X_1, X_2)$ is homotopy equivalent to $C_1 \times X_2 \smile X_1 \times C_2$.*

Proof. Since C_i and KX_i are contractible, the homotopy extension theorem implies that the inclusion maps $X_i \subset C_i$ and $X_i \subset KX_i$ may be extended to maps $r_i : KX_i \to C_i$ and $s_i : C_i \to KX_i$, $i = 1, 2$. These maps then induce

$$r : KX_1 \times X_2 \smile X_1 \times KX_2 \to C_1 \times X_2 \smile X_1 \times C_2$$

and

$$s : C_1 \times X_2 \smile X_1 \times C_2 \to KX_1 \times X_2 \smile X_1 \times KX_2$$

Since $KX_1 \times X_2 \smile X_1 \times KX_2$ is homeomorphic to $J(X_1, X_2)$, it suffices to show $sr \sim 1$ and $rs \sim 1$. We show the first; the second follows similarly. Define

$$h_i : KX_i \times 0 \smile KX_i \times 1 \smile X_i \times I \to KX_i \qquad i = 1, 2$$

by $h_i(\ , 1) = s_i r_i$, $h_i(\ , 0) = 1$ and $h_i(x_i, t) = x_i$ for $x_i \in X_i$, and $t \in I$. As above these maps can be extended to maps $KX_i \times I \to KX_i$ which in turn gives us a homotopy between sr and the identity.

We notice that we needed the fact that all spaces were countable CW-complexes and that the subspaces were embedded as subcomplexes to be able to apply the homotopy extension theorem.

THEOREM (2). *Let K_i be a contractible space containing X_i as a subcomplex, $i = 1, \ldots, n$. Then $J(X_1, \ldots, X_n)$ is homotopy equivalent to the subset*

$$\bigcup_{i=1}^{n} T_0(K_i, \ldots, K_{i-1}, X_i, K_{i+1}, \ldots, K_n) \text{ of } T_0(K_1, \ldots, K_n).$$

Proof. We show in Theorem (3) below that $J(X_1, \ldots, X_n)$ is homotopy equivalent to $Q(X_1, \ldots, X_n)$ (see below for the definition of Q). Using this fact, Theorem (2) is proven by an argument similar to that used in the proof of Theorem (1).

Definition (2). Let $j_i : T_0(C, \ldots, C, Id, C, \ldots, C) \to T_0 C$ be $T_0(1, \ldots, 1, i_C, 1, \ldots, 1)$ (i_C on the i^{th} co-ordinate). Q is defined to be $\bigcup_{i=1}^{n}$ Image j_i and is denoted by $Q(X_1, \ldots, X_n) = \bigcup_{i=1}^{n} T_0(CX_1, \ldots, X_i, \ldots, CX_n)$ with the obvious identifications. Since j_i is a natural transformation $T_0 C(f_1, \ldots, f_n)$ maps $Q(X_1, \ldots, X_n)$ to $Q(Y_1, \ldots, Y_n)$ and hence induces $Q(f_1, \ldots, f_n)$.

Remark. It is immediate that this definition is equivalent to the definition of Q given in §1.

THEOREM (3). *Q is homotopy equivalent to J ($Q, J : \mathscr{C}^n \to \mathscr{C}$).*

Proof. We shall prove the theorem by induction on n the "dimension" of the domain category. We first prove that $Q : \mathscr{C}^2 \to \mathscr{C}$ is homotopic to $J : \mathscr{C}^2 \to \mathscr{C}$. The map induced by $\pi \times \pi$, $KX_1 \times X_2 \smile X_1 \times KX_2 \to CX_1 \times X_2 \smile X_1 \times CX_2$ is a homotopy equivalence by Theorem (1). Let $H(X_1, X_2)$ be the composition of this map and the map, v, given following Definition (1). It is clear that H is a homotopy equivalence. Hence it suffices to check that H is natural. Let $f_i : X_i \to Y_i$, $i = 1, 2$. Then

$$\begin{aligned} H(Y_1, Y_2)J(f_1, f_2)(x_1, x_2, t) &= H(Y_1, Y_2)(f_1(x_1), f_2(x_2), t) \\ &= \begin{cases} (f_1(x_1), (f_2(x_2), 1 - 2t)) & 0 \le t \le \frac{1}{2} \\ ((f_1(x_1), 2t - 1), f_2(x_2)) & \frac{1}{2} \le t \le 1 \end{cases} \\ &= Q(f_1, f_2)\, H(X_1, X_2) \end{aligned}$$

We assume inductively that $J : \mathscr{C}^{n-1} \to \mathscr{C}$ is homotopy equivalent to $Q : \mathscr{C}^{n-1} \to \mathscr{C}$ and that this equivalence is given by a map H. Let $Q' : \mathscr{C}^n \to \mathscr{C}$ be defined by $Q'(X_1, \ldots, X_n) = Q(Q(X_1, \ldots, X_{n-1}), X_n)$ and $Q'(f_1, \ldots, f_n) = Q(Q(f_1, \ldots, f_{n-1}), f_n)$. Q' is clearly a functor. The proof of the inductive step consists of showing that Q' is homotopy equivalent to both J and Q. The composition of the two homotopy equivalences then gives us the map $H : J \to Q$.

$H' : J \to Q'$ is given by setting $H'(X_1, \ldots, X_n)$ equal to the composition

$$H(Q(X_1, \ldots, X_{n-1}), X_n)J(H(X_1, \ldots, X_{n-1}, 1).$$

It is clear by induction and Theorem (1) that H' is a homotopy equivalence. We therefore check that H' is natural.

$$\begin{aligned} &H'(Y_1, \ldots Y_n)J(f_1, \ldots, f_n) \\ &\quad = H(Q(Y_1, \ldots, Y_{n-1}), Y_n)J(H(Y_1, \ldots, Y_{n-1}), 1)J(J(f_1, \ldots, f_{n-1}), f_n) \\ &\quad = H(Q(Y_1, \ldots, Y_{n-1}), Y_n)J(H(Y_1, \ldots, Y_{n-1})J(f_1, \ldots, f_{n-1}), f_n) \\ &\quad = H(Q(Y_1, \ldots, Y_{n-1}), Y_n)J(Q(f_1, \ldots, f_{n-1})H(X_1, \ldots, X_{n-1}), f_n) \\ &\quad = H(Q(Y_1, \ldots, Y_{n-1}), Y_n)J(Q(f_1, \ldots, f_{n-1}), f_n)J(H(X_1, \ldots, X_{n-1}), 1) \\ &\quad = Q(Q(f_1, \ldots, f_{n-1}), f_n)H(Q(X_1, \ldots, X_{n-1}), X_n)J(H(X_1, \ldots, X_{n-1}), 1) \\ &\quad = Q'(f_1, \ldots, f_n)H'(X_1, \ldots, X_n). \end{aligned}$$

We now define $R : Q' \to Q$. Let r extend the inclusion

$$Q(X_1, \ldots, X_{n-1}) \to T_0C(X_1, \ldots, X_{n-1}) \text{ to } CQ(X_1, \ldots, X_{n-1}).$$

We may define, as in the proof of Theorem (1), a homotopy h between $rCQ(f_1, \ldots, f_{n-1})$ and $T_0C(f_1, \ldots, f_{n-1})r$ which is $Q(f_1, \ldots, f_{n-1})$ on $Q(X_1, \ldots, X_{n-1}) \times I$. $R(X_1, \ldots, X_n)$ is defined to be $r \times 1$ on $CQ(X_1, \ldots, X_{n-1}) \times X_n$ and the identity on $Q(X_1, \ldots, X_{n-1}) \times CX_n$. This, as was shown in Theorem (1), is a homotopy equivalence. Moreover h may be extended to a homotopy between the maps

$$R(Y_1, \ldots, Y_n)Q'(f_1, \ldots, f_n) \text{ and } Q(f_1, \ldots f_n)R(X_1, \ldots, X_n).$$

The composition $RH' = H$ is the desired homotopy equivalence between J and Q. This completes the proof of the theorem.

Arkowitz [1, p. 11] shows that $J(X_1, X_2)$ is homotopy equivalent to $\Sigma\wedge(X_1, X_2)$ (see §1 for the definition of $\Sigma\wedge$). Such an equivalence is given by

$$\begin{aligned} H(X_1, X_2)(x_1, x_2, t) &= (t, (x_1, x_2)) \\ H(Y_1, Y_2)J(f_1, f_2)(x_1, x_2, t) &= H(Y_1, Y_2)(f_1(x_1), f_2(x_2), t) \\ &= (t, (f_1(x_1), f_2(x_2)) \\ &= \Sigma\wedge(f_1, f_2)H(X_1, X_2)(x_1, x_2, t) \end{aligned}$$

Hence $J : \mathscr{C}^2 \to \mathscr{C}$ and $\Sigma\wedge : \mathscr{C}^2 \to \mathscr{C}$ are homotopy equivalent.

THEOREM (4). *$\Sigma^{n-1}\wedge$ and J are homotopy equivalent functors from the category $\mathscr{C}^n$ to $\mathscr{C}$.*

Proof. The above paragraph is the proof for the case $n = 2$. We assume inductively that $\Sigma^{n-2}\wedge$ and J are homotopy equivalent and that this equivalence is given by H.

We first remark that it is well known that the functors $\vee(\vee \times 1)$ and $\vee : \mathscr{C}^n \to \mathscr{C}$ are homotopy equivalent and that the functors $\Sigma\wedge$, $\wedge T_0(\Sigma, 1)$ and $\wedge T_0(1, \Sigma)$ are all homotopy equivalent functors from the category $\mathscr{C}^2$ to $\mathscr{C}$.

Let $H : J \to \Sigma^{n-1}\wedge$ be given by

$$H(X_1, \ldots, X_n) = H(\Sigma^{n-2}\wedge(X_1, \ldots, X_{n-1}), X_n)J(H(X_1, \ldots, X_{n-1}), 1).$$

Using the induction hypothesis and the above remark a routine argument shows that H is a homotopy equivalent transformation $\Sigma^{n-1}\wedge \to J$.

COROLLARY. *The functors $\Sigma^{n-1}\wedge$ and Q are homotopy equivalent. Hence there exists a homotopy equivalent transformation $\bar{h} : \Sigma^{n-1}\wedge \to Q$.*

This proves Theorem (1.2).

REFERENCES

1. M. ARKOWITZ: The generalized Whitehead product, *Pacific J. Math.* **12** (1962), 7–23.
2. A. L. BLAKERS and W. S. MASSEY: Products in homotopy theory, *Ann. Math., Princeton* **58** (1953), 295–324.
3. A. BOREL and J-P. SERRE: Groupes de Lie et puissances reduites de Steenrod, *Amer. J. Math.* **75** (1953), 409–448.
4. K. A. HARDIE: Higher Whitehead products, *Quart. J. Math.* **12** (1961), 241–249.
5. K. A. HARDIE: On a construction of E. C. Zeeman, *J. Lond. Math. Soc.* **35** (1960), 452–464.
6. P. J. HILTON: Homotopy theory and duality, Mimeographed notes, Cornell University (1959).
7. S-T. HU: *Homotopy Theory*, Academic Press, New York 1959.
8. I. M. JAMES: Reduced product spaces, *Ann. Math., Princeton* **62** (1955), 170–197.
9. I. M. JAMES: On Lie groups and their homotopy groups, *Proc. Camb. Phil. Soc.* **55** (1959), 244–247.
10. J-P. SERRE: Groupes d'homotopie et classes de groupes abéliens, *Ann. Math., Princeton* **58** (1953), 258–294.
11. J. H. C. WHITEHEAD: On adding relations to homotopy groups, *Ann. Math., Princeton* **42** (1941), 409–428.

Cornell University and
Massachusetts Institute of Technology